Lecture 2 - Path Integrals

The path integral provides an alternate formulation of QFT. Believed to be equivallent to canonical quantization.

Facilitates quantization of much broader class of theories, e.g. those with derivative interactions such as non-Abelian gauge theory.

Moments of the path integral are time ordered correlation functions

$$\int \mathcal{D}\varphi \, \varphi(x_1) \ldots \varphi(x_n) \, e^{iS[\varphi]} = \langle T \hat{\phi}(x_1) \ldots \hat{\phi}(x_n) \rangle$$

These objects can be converted to S-matrix elements using procedure known as LSZ-reduction.

Path integrals are well defined in Eucledian space where $e^{iS} \to e^{-S_E}$.

The Minkowski space path integral is defined by analytic continuation.

Quick primer on Gaussian integration:

Want to evaluate integrals of the form

$$I = \int_{-\infty}^{\infty} dp \, \exp\left[-\tfrac{1}{2} a p^2 + Jp\right]$$

First complete the square

$$I = \int dp \, \exp\left[-\tfrac{1}{2} a \left(p - \tfrac{J}{a}\right)^2 + \frac{J^2}{2a}\right]$$

Shift $p \to p + \frac{J}{a}$. Measure unchanged

$$\Rightarrow I = \exp\left(J^2/2a\right) \int_{-\infty}^{\infty} dp \, \exp\left[-\tfrac{1}{2} a p^2\right] = \frac{\exp\left(J^2/2a\right)}{\sqrt{a}} \int dp \, e^{-p^2/2}$$

Use following trick:

$$\left[\int dp \, e^{-p^2/2}\right]^2 = \int dx \int dy \, e^{-x^2/2} e^{-y^2/2} = 2\pi \int_0^{\infty} r dr \, e^{-r^2/2}$$

$$= \pi \int_0^{\infty} dr^2 \, e^{-r^2/2} = 2\pi$$

$$\Rightarrow I = \sqrt{\frac{2\pi}{a}} \exp\left[J^2/2a\right]$$

Matrix exponential

$$\exp \hat{M} \equiv \sum_{k=0}^{\infty} \frac{1}{k!} \hat{M}^k \quad \text{(Taylor Expansion)}$$

Finally, generalize $a p^2 \to p_i^* a_{ij} p_j = \vec{p}^{\dagger} \hat{A} \vec{p}$ ($\hat{A}$ is matrix)

Diagonalizing $\hat{A} \Rightarrow$ product of 1-D Gaussian integrals w/

$a \to$ eigenvalues of $\hat{A} \Rightarrow$ prod of eigenvalues $(\hat{A}) = \det \hat{A}$

$$\Rightarrow \int_{-\infty}^{\infty} d\vec{p} \, \exp\left[-\vec{p}^{\dagger} \hat{A} \vec{p}/2 + \vec{J}^{\dagger} \vec{p}\right] = \sqrt{\frac{(2\pi)^n}{\det \hat{A}}} \exp\left[\vec{J}^{\dagger} \hat{A}^{-1} \vec{J}/2\right]$$

Review of Path Integrals in QM

Let $\hat{H} = \frac{1}{2m}\hat{P}^2 + V(\hat{Q})$ w/ $[\hat{P}, \hat{Q}] = i$

To begin, compute propagator $\langle q'', t'' | q', t' \rangle = \langle q'' | e^{-iH\Delta t} | q' \rangle$

w/ $|q, t\rangle = e^{iHt}|q\rangle$

To evaluate, divide time interval $T \equiv t'' - t'$ into $N+1$ pieces w/ $\delta t = T/(N+1)$ and introduce N sets of position eigenstates

$$\langle q'', t'' | q', t' \rangle = \int \prod_{j=1}^{N} dq_j \, \langle q'' | e^{-iH\delta t} | q_N \rangle \langle q_N | e^{-iH\delta t} | q_{N-1} \rangle$$

$$\cdots \langle q_1 | e^{-iH\delta t} | q' \rangle$$

Consider e.g. $\langle q_2 | e^{-iH\delta t} | q_1 \rangle$

Recall, Campbell-Baker-Hausdorf

$$\exp(A+B) = \exp(A)\exp(B)\exp\left(-\tfrac{1}{2}[A,B] + \cdots\right)$$

Implies $\exp(-iH\delta t) = \exp\left[-i(\delta t/2m) P^2\right] \exp\left[-i\delta t\, V(Q)\right] \exp\left[\mathcal{O}(\delta t^2)\right]$

$$\Rightarrow \langle q_2 | e^{-iH\delta t} | q_1 \rangle = \int \frac{dp_1}{2\pi} \langle q_2 | e^{-i(\delta t/2m)P^2} | p_1 \rangle \langle p_1 | e^{-i\delta t V(Q)} | q_1 \rangle$$

$$= \int \frac{dp_1}{2\pi} e^{-i(\delta t/2m)p_1^2} e^{-i\delta t V(q_1)} \langle q_2 | p_1 \rangle \overbrace{\langle p_1 | q_2 \rangle}^{\exp(-ip_1 q_2)}$$

$$= \int \frac{dp_1}{2\pi} e^{-iH(p_1, q_1)\delta t} e^{ip_1(q_2 - q_1)}$$

(Subtelty about ordering. Can use Weyl ordering, see Srednicki p 44)

$$\Rightarrow \langle q'',t'' | q',t' \rangle = \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j (q_{j+1} - q_j)} e^{-iH(p_j,q_j)\delta t}$$

w/ $q_0 = q'$ & $q_{N+1} = q''$

Finally introduce some notation (Mention normalization)

$$\int_{q'}^{q''} \mathcal{D} q = \lim_{N\to\infty} \int dq_1 \int dq_{N-1} \left(\frac{m}{2\pi i \delta t} \right)^{N/2}$$

which is a functional integral over all paths w/ end points X_0 & X'

w/ similar for $\mathcal{D}p$

$$\Rightarrow \langle q'', t'' | q', t' \rangle = \int \mathcal{D} q \, \mathcal{D} p \exp\left[i \int_{t'}^{t''} dt \left(p(t) \dot{q}(t) - H(p(t), q(t)) \right) \right]$$

Note if $H = \frac{P^2}{2m} + V(Q)$ then p-integrals are Gaussian

$$\int \frac{dp_i}{2\pi} \exp\left[-i \frac{P_i^2 \delta t}{2m} + i P_i (q_i - q_{i-1}) \right] = \sqrt{\frac{m}{2\pi i \delta t}} \exp\left[i \left(\frac{m \delta t}{2} \right) \left(\frac{q_i - q_{i-1}}{\delta t} \right)^2 \right]$$

$$\Rightarrow \langle q'', t'' | q', t' \rangle = \lim_{N\to\infty} \int dq_1 \int dq_N \left(\frac{m}{2\pi i \delta t} \right)^{N/2} \exp\left\{ i \delta t \sum_{i=1}^{N} \left[\frac{m}{2} \left(\frac{q_i - q_{i-1}}{\delta t} \right)^2 - V(q_{i-1}) \right] \right\}$$

Then as $N \to \infty$ and $\Delta t \to dt \Rightarrow$

$$\lim i \delta t \sum_{i=1}^{N} \left[\frac{m}{2} \left(\frac{q_i - q_{i-1}}{\delta t} \right)^2 - V(q_{i-1}) \right] \Big\} = i \int_{t'}^{t''} dt \, L(q, \dot{q})$$

w/ $L(q, \dot{q}) = \frac{m}{2} \left(\frac{dq}{dt} \right)^2 - V(q) = \frac{1}{2} m \dot{q}^2 - V(q)$ is the Lagrangian

$$\Rightarrow \langle q'' t'' | q' t' \rangle = \int_{x'}^{x''} \mathcal{D}q \; e^{iS}$$

w/ $S = \int_{t'}^{t''} dt \, L(q, \dot{q})$ is the action

and $L(\dot{q}, q)$ is computed by finding the stationary point of the p integral by solving

$$0 = \frac{\partial}{\partial p}\left(p\dot{q} - H(p,q)\right) = \dot{q} - \frac{\partial H(p,q)}{\partial p}$$ (we will generalize this for fields)

Next, let's compute an expectation value (no assumption of P dependence)

$$\langle q'', t'' | Q(t_1) | q', t' \rangle = \langle q'' | e^{-iH(t''-t_1)} \, Q \, e^{-iH(t_1 - t')} | q' \rangle$$

Inserting Q at time t_1 w/ $t' < t_1 < t''$, yields factor of $q(t_1)$

$$\Rightarrow \langle q'', t'' | Q(t_1) | q', t' \rangle = \int \mathcal{D}p \, \mathcal{D}q \; q(t_1) \, e^{iS}$$

w/ $S = \int_{t'}^{t''} dt \, (p\dot{q} - H)$

Next, let's generalize to two point function. Have two operators $Q(t_1)$ & $Q(t_2)$

Clearly their order must depend on $t_1 < t_2$ or $t_2 < t_1$ when inserting $\mathbb{1}$

$$\Rightarrow \int \mathcal{D}p \, \mathcal{D}q \; q(t_1) \, q(t_2) \, e^{iS} = \langle q'', t'' | T \, Q(t_1) \, Q(t_2) | q' t' \rangle$$

Need functional derivative.

Generalize $\frac{\partial}{\partial x_i} x_j = \delta_{ij} \Rightarrow \frac{\delta}{\delta f(t_1)} f(t_2) = \delta(t_1 - t_2)$

Has all same rules as derivatives (product, chain, etc)

Modify Hamiltonian to include classical sources (will set to zero at end)

$$H(p,q) \to H(p,q) - f(t)\,q(t) - h(t)\,p(t)$$

for some classical source $f(t)$ & $h(t)$

$$\Rightarrow \langle q'', t'' | q', t' \rangle_{f,h} = \int \mathcal{D}q\,\mathcal{D}p\, \exp\Big[i \underbrace{\int_{t'}^{t''} dt\, (p\dot{q} - H + fq + hp)}_{S_{f,h}} \Big]$$

(H is original Hamiltonian)

$$\Rightarrow \frac{1}{i}\frac{\delta}{\delta f(t_1)} \langle q'', t'' | q', t' \rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\; q(t_1)\, e^{iS_{f,h}}$$

$$\frac{1}{i}\frac{\delta}{\delta h(t_1)} \langle q'', t'' | q', t' \rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\; p(t_1)\, e^{iS_{f,h}}$$

$$\frac{1}{i^2}\frac{\delta}{\delta f(t_1)}\frac{\delta}{\delta f(t_2)} \langle q'', t'' | q', t' \rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\; q(t_1)\,q(t_2)\, e^{iS_{f,h}}$$

Then at the end, set $f = h = 0$

$$\Rightarrow \langle q'', t'' | T\, Q(t_1) \ldots P(t_n) \ldots | q', t' \rangle$$

$$= \frac{1}{i}\frac{\delta}{\delta f(t_1)} \cdots \frac{1}{i}\frac{\delta}{\delta h(t_n)} \cdots \langle q'', t'' | q', t' \rangle_{f,h} \Big|_{f=h=0}$$

LSZ tells us we want vacuum matrix elements

$\Rightarrow$ need to take $t' \to -\infty$ and $t'' \to +\infty$

Note: $$\langle 0 | 0 \rangle_{f,h} = \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} \int dq''\, dq'\; \psi_0^*(q'')\, \langle q'', t'' | q', t' \rangle_{f,h}\, \psi_0(q')$$

w/ $\psi_0(q) = \langle q | 0 \rangle$

Let $H|n\rangle = E_n|n\rangle$ and $E_0 = 0$ (can shift zero point energy to make this true)

$$\Rightarrow |q',t'\rangle = e^{iHt'}|q'\rangle = \sum_{n=0}^{\infty} e^{iHt'}|n\rangle\langle n|q'\rangle$$

$$= \sum \psi_n^*(q')\, e^{iE_nt'}|n\rangle \qquad w/\ \psi_n(q) = \langle q|n\rangle$$

Next, regulate limit w/ $H \to H(1-i\varepsilon)$

$\Rightarrow$ as $t' \to -\infty$, only ground state survives. Same for $t'' \to \infty$.

$\Rightarrow$ Limit returns $\psi_0^*(q')|0\rangle$

Then introduce $|\chi\rangle$ s.t. $\langle 0|\chi\rangle \neq 0 = \chi(q')$

Integrate over $q' \Rightarrow$ constant $\Rightarrow$ absorb in normalization.

Same for q''

$$\Rightarrow \langle 0|0\rangle_{f,h} = \int \mathcal{D}p\,\mathcal{D}q\ \exp\left[i\int_{-\infty}^{\infty} dt\left(p\dot{q} - (1-i\varepsilon)H + fq + hp\right)\right]$$

Then assuming quadratic P dependence $\Rightarrow$ Move to Lagrangian

If we are doing perturbation thy, separate $L = L_0 + L_I$

$$\Rightarrow \langle 0|0\rangle_f = \int \mathcal{D}q\ \exp\left[i\int_{-\infty}^{\infty}\left(L_0 + L_I + fq\right)\right] \qquad (H_I = -L_I)$$

$$= \exp\left[i\int_{-\infty}^{\infty} dt\ L_I\left(\frac{1}{i}\frac{\delta}{\delta f(t)}\right)\right] \times$$

will treat this factor perturbatively $\Rightarrow$ Feynman Rules!

$$\int \mathcal{D}q\ \exp\left[i\int_{-\infty}^{\infty}\left(L_0(\dot{q},q) + fq\right)\right]$$

Path Integral for QM SHO (no interactions)

SHO : $\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2 \hat{Q}^2$

Quadratic $\hat{P}$ dependence $\Rightarrow$ can integrate $\mathcal{D}p$ and work w/ L

Note Let $m^{-1} \to (1-i\varepsilon)m^{-1}$ (or $m \to (1+i\varepsilon)m$)

and $m\omega^2 \to (1-i\varepsilon)m\omega^2 \Rightarrow \hat{H} \to \hat{H}(1-i\varepsilon)$

$$\Rightarrow \langle 0|0\rangle_f = \int \mathcal{D}q \, \exp\left[i\int dt \left(\tfrac{1}{2}(1+i\varepsilon)m\dot{q}^2 - \tfrac{1}{2}(1-i\varepsilon)m\omega^2 q + fq\right)\right]$$

For notational simplicity, take $m=1$

Introduce Fourier Trans variable

$$\tilde{q}(E) = \int_{-\infty}^{\infty} dt\, e^{iEt} q(t) \iff q(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iEt}\tilde{q}(E)$$

$$\Rightarrow \left(\tfrac{1}{2}(1+i\varepsilon)m\dot{q}^2 - \tfrac{1}{2}(1-i\varepsilon)m\omega^2 q + fq\right) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dE}{2\pi}\frac{dE'}{2\pi} e^{-i(E+E')t}$$

$$\times\left[\left(-(1+i\varepsilon)EE' - (1-i\varepsilon)\omega^2\right)\tilde{q}(E)\tilde{q}(E') + \tilde{f}(E)\tilde{q}(E') + \tilde{f}(E')\tilde{q}(E)\right]$$

Since all t dependence is in prefactor, t-integral $\Rightarrow 2\pi\delta(E+E')$

Then integrate over $E' \Rightarrow$

$$S = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dE}{2\pi}\Big[\left((1+i\varepsilon)E^2 - (1-i\varepsilon)\omega^2\right)\tilde{q}(E)\tilde{q}(-E) + \tilde{f}(E)\tilde{q}(-E) + \tilde{f}(-E)\tilde{q}(E)\Big]$$

redefine ε

Note $(1+i\varepsilon)E^2 - (1-i\varepsilon)\omega^2 = E^2 - \omega^2 + i(E^2+\omega^2)\varepsilon \to E^2 - \omega^2 + i\varepsilon$

Change integration variables:

$$\tilde{x}(E) = \tilde{q}(E) + \frac{\tilde{f}(E)}{E^2-\omega^2+i\varepsilon} \Rightarrow \mathcal{D}q \to \mathcal{D}x \quad \text{(shift is by constant in } q\text{)}$$

$$\Rightarrow S' = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dE}{2\pi}\left[\tilde{x}(E)(E^2-\omega^2+i\varepsilon)\tilde{x}(-E) - \frac{\tilde{f}(E)\tilde{f}(-E)}{E^2-\omega^2+i\varepsilon}\right]$$

$$\Rightarrow \langle 0|0\rangle_f = \exp\left[\frac{-i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{\tilde{f}(E)\tilde{f}(-E)}{E^2-\omega^2+i\varepsilon}\right] \times$$

$$\int \mathcal{D}x \exp\left[\frac{i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\tilde{x}(E)(E^2-\omega^2+i\varepsilon)\tilde{x}(-E)\right]$$

If we set external force to zero $(f=0) \Rightarrow$ system in ground state will stay in ground state $\Rightarrow$

$$\langle 0|0\rangle_{f=0} = 1 = \int \mathcal{D}x[\ \ ]$$

$$\Rightarrow \langle 0|0\rangle_f = \exp\left[\frac{-i}{2}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{\tilde{f}(E)\tilde{f}(-E)}{E^2-\omega^2+i\varepsilon}\right]$$

Fourier transforming back

contour integration

$$\Rightarrow G(t-t') = \int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{-e^{-iE(t-t')}}{E^2-\omega^2+i\varepsilon} = \frac{i}{2\omega}\exp(-i\omega|t-t'|)$$

is Green's fn $\left(\frac{\partial^2}{\partial t^2}+\omega^2\right)G(t-t') = \delta(t-t')$

$$\Rightarrow \langle 0|0\rangle_f = \exp\left[\frac{i}{2}\int_{-\infty}^{\infty} dt\, dt'\, f(t)\, G(t-t')\, f(t')\right]$$

Compute time ordered vacuum expectation values

$$\langle 0|TQ(t_1)\dots|0\rangle = \frac{1}{i}\frac{\delta}{\delta f(t_1)}\cdots\langle 0|0\rangle_f\Big|_{f=0}$$

2-pt: $$\langle 0|TQ(t_1)\,Q(t_2)|0\rangle = \frac{1}{i}\frac{\delta}{\delta f(t_1)}\frac{1}{i}\frac{\delta}{\delta f(t_2)}\langle 0|0\rangle_f\Big|_{f=0}$$

$$= \frac{1}{i}\frac{\delta}{\delta f(t_1)}\left[\int_{-\infty}^{\infty}dt'\,G(t_2-t')f(t')\right]\langle 0|0\rangle_f\Big|_{f=0}$$

$$= \frac{1}{i}G(t_2-t_1) + (\text{term w/ } f\text{'s})\Big]\langle 0|0\rangle_f\Big|_{f=0}$$

$$= \frac{1}{i}G(t_2-t_1) \quad \text{as expected}$$

3-pt: If # of Q's is odd, always left w/ $f \Rightarrow 0$

4-pt: $\langle 0|TQ(t_1)\,Q(t_2)Q(t_3)\,Q(t_4)|0\rangle$

$$= \frac{1}{i^2}\left[G(1-2)G(3-4)+G(1-3)G(2-4)+G(1-4)G(2-3)\right]$$

2n-pt: $\langle 0|TQ(t_1)\dots Q(t_{2n})|0\rangle$

$$= \frac{1}{i^n}\sum_{\text{pairings}} G(t_{i_1}-t_{i_2})\dots G(t_{i_{2n-1}}-t_{i_{2n}})$$

PI encodes the Euler-Lagrange equation

Shift the trajectory $x(t)$ by small amount
$x(t) \to x(t) + \delta x(t)$ w/ bc fixed $\quad \delta x(t') = \delta x(t_0) = 0$

This is just a change of variables $\Rightarrow$ PI is invariant under shift

$$\int \mathcal{D}x \, e^{iS[x(t)+\delta x(t)]} = \int \mathcal{D}x \, e^{iS[x(t)]} \Rightarrow$$

$$0 = \int \mathcal{D}x \, e^{iS[x(t)+\delta x(t)]} - \int \mathcal{D}x \, e^{iS[x(t)]} = \int \mathcal{D}x \, e^{iS[x(t)]} \, i\delta S$$

From classical Lagrangian mechanics, we know

$$\delta S = S[x(t)+\delta x(t)] - S[x(t)] = \int_{t_i}^{t_f} dt \left(\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x} \right)$$

$$= \underbrace{\cancel{\frac{\partial L}{\partial \dot{x}}\delta x \Big|_{x_0}^{x'}}}_{0} + \int_{t_0}^{t'} dt \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right)\delta x = 0$$

since $\delta x = 0$ at boundaries

classical least action principle

$$\Rightarrow \int \mathcal{D}[x(t)] e^{iS[x(t)]} \quad i\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right) = i\left\langle \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right\rangle = 0$$

This is a manifestation of Ehrenfest's Theorem $\Rightarrow$ expectation values follow classical trajectories

The Classical Limit

What distinguishes classical path? Classical path is minimum of the action

Take small perturbation around classical path parameterized by ε

$x = \frac{x'}{t'}\left[t + \varepsilon \frac{t(t-t')}{t'}\right]$ which still satisfies $x(0)=0$ and $x(t')=x'$

Then $S = \int_0^{t'} dt \, \frac{m\dot{x}^2}{2} = \int_0^{t} dt \, \frac{m}{2}\left(\frac{x'}{t'}\right)^2\left[1 + \varepsilon\frac{(2t-t')}{t'}\right]^2 = \frac{m}{2}\frac{x'^2}{t'}\left(1+\frac{\varepsilon^2}{3}\right) = S_{cl}\left(1+\frac{\varepsilon^2}{3}\right)$

$\Rightarrow$ No linear term in ε. Had to be this way since $\delta S_{cl} = \left(\frac{\partial S_{cl}}{\partial \varepsilon}\right)\Big|_{\varepsilon=0} \varepsilon = 0$ at min

$\Rightarrow$ To leading order $e^{i(S_{cl}+\delta S)/\hbar} = e^{iS_{cl}/\hbar} + \mathcal{O}(\varepsilon^2) \Rightarrow$ All paths near classical one have roughly same phase & add

Let's do similar thing w/ non-classical path $\quad x = \frac{x'}{t'^2}\left[t + \varepsilon\frac{t(t-t')}{t'}\right]^2$

and again $x(0)=0$ and $x(t')=x'$

$S = \frac{2mx'^2}{3t'}\left(1 + \frac{\varepsilon}{2} + \mathcal{O}(\varepsilon^2)\right) = S_{ct}\left(1 + \frac{\varepsilon}{2} + \quad\right)$ so $\delta S = \frac{\partial S}{\partial \varepsilon}\Big|_{\varepsilon=0} \varepsilon \neq 0$ as expected

Then change in phase is 1^{st} order and cancellations can happen between paths w/ $\varepsilon > 0$ and $\varepsilon < 0$.

Path Integral in QFT & Feynman Rules

Generalize SHO to QFT:

$q(t) \to \phi(\vec{x},t)$ (classical field) (set $\epsilon \to 0$ for simplicity)

$Q(t) \to \varphi(\vec{x},t)$ (operator field)

$f(t) \to J(\vec{x},t)$ (classical source)

Path Integral free-theory: $\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$

$$Z_0[J] \equiv \langle 0|0\rangle_J = \int \mathcal{D}\phi \, \exp\left[i\int d^4x\left(\mathcal{L}_0 + J\phi\right)\right]$$

w/ $\mathcal{D}\phi \propto \prod_x d\phi(x)$

Integrate over space of field configurations

Follow same steps as for SHO:

FT the field $\tilde{\phi}(k) = \int d^4x \, e^{-ikx}\phi(x)$ and $\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx}\tilde{\phi}(k)$

Change variables: $\tilde{\chi}(k) = \tilde{\phi}(k) + \frac{\tilde{J}}{k^2-m^2} \Rightarrow \mathcal{D}\phi = \mathcal{D}\chi$

$$\Rightarrow S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left[\frac{-\tilde{J}(k)\tilde{J}(-k)}{k^2-m^2} + \underbrace{\tilde{\chi}(k)\left(k^2-m^2\right)\chi(-k)}\right]$$

integrate over $\mathcal{D}\chi$ $\Rightarrow Z_0[0] = \langle 0|0\rangle\big|_{J=0} = 1$

absorb i into D_F

$$\Rightarrow Z_0[J] = \exp\left[-\int d^4x \int d^4y \, \frac{1}{2} J(x) D_F(x-y) J(y)\right]$$

Feynman propagator

2.14

Satisfies $(\Box_x + m^2) D_F(x,y) = -i\delta^4(x-y)$

Then $\langle 0 | T \hat{\varphi}_0(x) \hat{\varphi}_0(y) | 0 \rangle = \frac{1}{i^2} \frac{\delta^2}{\delta J(x) \delta J(y)} \Big|_{J=0} = D_F(x,y)$

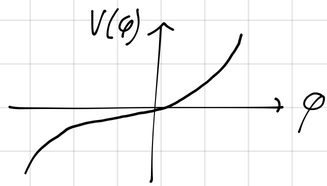
Odd time ordered products vanish

$$\langle 0 | T \hat{\varphi}_0(x_1) \hat{\varphi}_0(x_2) \hat{\varphi}_0(x_3) \hat{\varphi}_0(x_4) | 0 \rangle = D_F(1,2) D_F(3,4) + D_F(1,3) D_F(2,4) + D_F(1,4) D_F(2,3)$$

Interacting theory

Ex: $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 + \frac{1}{6} g \varphi^3 + Y\varphi$

Note: potential has shape



$\Rightarrow$ unstable! (roll down the hill)

But can be ignored in perturbation thy $\Rightarrow$ ignore this

Goal: evaluate path integral for interacting theory

$$Z[J] \equiv \langle 0 | 0 \rangle_J = \int \mathcal{D}\varphi \, \exp\left(i \int d^4x \left(\mathcal{L}_0 + \mathcal{L}_1 + J\varphi\right)\right)$$

Use same steps as for SHO:

$$Z[J] = \exp\left(i \int d^4x \, \mathcal{L}_1\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right) \int \mathcal{D}\varphi \, \exp\left(i \int d^4x \left(\mathcal{L}_0 + J\varphi\right)\right)$$

$$\propto \exp\left(i \int d^4x \, \mathcal{L}_1\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right) Z_0[J]$$

w/ $Z_0[J] = \exp\left(\frac{-1}{2} \int d^4x \, d^4x' \, J(x) D_F(x-x') J(x')\right)$

Want to enforce (order by order in the g expansion) $\boxed{2.15}$

$\langle\Omega|\varphi(x)|\Omega\rangle = 0$ and $\langle k|\varphi(x)|0\rangle = e^{-ikx}$

Seperate $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I + \mathcal{L}_{CT}$

$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\varphi)^2 - m^2\varphi^2$

$\mathcal{L}_I = \frac{1}{6}g\varphi^3$

$\mathcal{L}_{CT} = Y\varphi$ + others w/ $\mathcal{L}_{CT} \sim \mathcal{O}(g)$ "counter terms" Z_i in text

Expand $Z_1[J] \propto \exp\left[\frac{i}{6}g\int d^4x\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right] Z_0[J]$

Where normalization is fixed by requiring $Z_1[0] = 1$

Dual Taylor expand in g and J

Note: overall factor $\frac{1}{V!}\,\frac{1}{6^V}\,\frac{1}{P!}\,\frac{1}{2^P}$

$$\Rightarrow Z_1[J] \propto \sum_{V=0}^{\infty}\frac{1}{V!}\left[\frac{ig}{6}\int d^4x\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right]^V$$

$$\times\sum_{P=0}^{\infty}\frac{1}{P!}\left[\frac{-1}{2}\int d^4y\, d^4z\, J(y)\, D_F(y-z)\, J(z)\right]^P$$

Note: E = # surviving sources (after acting w/ $\delta/\delta J$) (also stands for external)

V = # vertices $\qquad P$ = # propagators

Then $E = 2P - 3V$

Overall phase $i^{-V}\left(\frac{1}{i}\right)^{3V} i^{-P} = i^{-V+E-P}$ (recall i in D_F) 2.16

and $3V$ functional derivatives act on $2P$ sources $\frac{2P!}{(2P-3V)!}$ combos

Note: many expressions are algebraically identical

To organize terms, introduce Feynman diagrams:

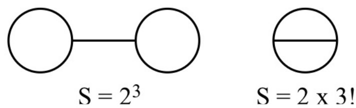


Figure 9.1: All connected diagrams with $E = 0$ and $V = 2$.

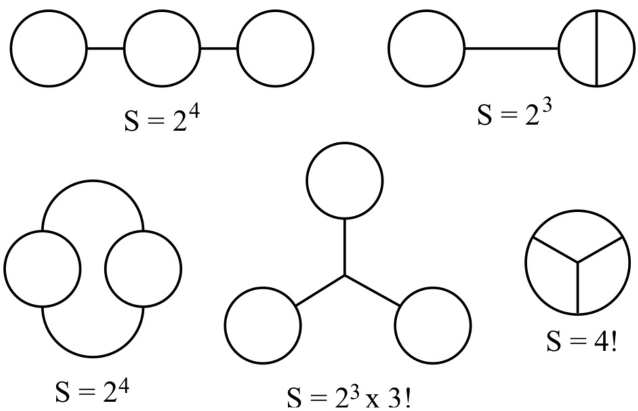


Figure 9.2: All connected diagrams with $E = 0$ and $V = 4$.

2.17

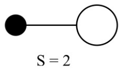


Figure 9.3: All connected diagrams with $E = 1$ and $V = 1$.

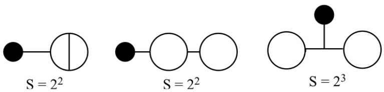


Figure 9.4: All connected diagrams with $E = 1$ and $V = 3$.

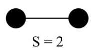


Figure 9.5: All connected diagrams with $E = 2$ and $V = 0$.

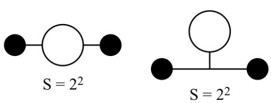


Figure 9.6: All connected diagrams with $E = 2$ and $V = 2$.

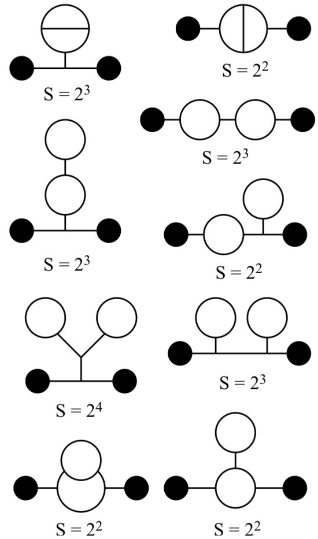


Figure 9.7: All connected diagrams with $E = 2$ and $V = 4$.

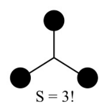


Figure 9.8: All connected diagrams with $E = 3$ and $V = 1$.

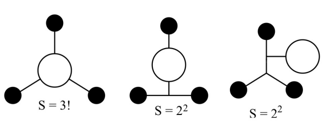


Figure 9.9: All connected diagrams with $E = 3$ and $V = 3$.

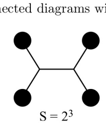


Figure 9.10: All connected diagrams with $E = 4$ and $V = 2$.

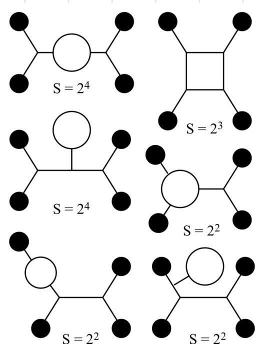


Figure 9.11: All connected diagrams with $E = 4$ and $V = 4$.

Reading Feynman diagrams:

- line segment is a propagator D_F
- Filled circle is a source $i\int d^4x\, J(x)$
- Three line together is vertex $ig\int d^4x$

"Connected": trace continuous path connecting any two points

Symmetry factor: S is factor by which diagram is overcounted.

First count terms resulting from Taylor expansion and compare to diagram:

- Can rearrange three derivatives from a particular vertex
  $\Rightarrow 3!$ for each vertex

- Can rearrange vertices $\Rightarrow V!$

- Can rearrange sources at end of propagator $\Rightarrow 2!$ for each prop

- Can rearrange propagators $\Rightarrow P!$

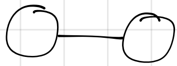
$\Rightarrow$ total factor $(3!)^V\, V!\, (2!)^P\, P!$

$\Rightarrow$ cancels $\frac{1}{V!}\frac{1}{6^V}\frac{1}{P!}\frac{1}{2^P}$ factor from Taylor expansions.

Where does overcounting come from?

When some rearrangement of derivatives gives same match-up to sources as some rearrangement of sources

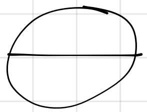
2.19

Ex: 

Flip both loops $\Rightarrow 2$ for each
Flip prop and vertices $\Rightarrow 2$

$$\Rightarrow S = 2^3$$

Ex:



rearrange propagators and exchange vertices
$\Rightarrow 3!$
Swap prop endpoints and vertices $\Rightarrow 2$

$$\Rightarrow S = 12$$

$Z[J]$ includes both connected and disconnected diagrams

<u>Let C_I be</u> a connected diagram (including symmetry factor)

General diagram $D = \frac{1}{S_D} \prod_I (C_I)^{n_I}$

w/ n_I counts number of C_I's in D and S_D is additional symmetry factor

Determine S_D: Only need exchanges of props and vertices between different connected diagrams

Only leaves D unchanged if
- exchange made between different, identical connected diagrams
- exchange involves all props + vertices in diagram

$\Rightarrow$ If n_I factors of C_I in $D \Rightarrow n!$ rearrangements

$$\Rightarrow S_D = \prod_I n_I!$$

Up to norm: (each D labeled by n_I)

$$Z_1[J] \propto \sum_{\{n_I\}} D \propto \sum_{\{n_I\}} \prod_I \frac{1}{n_I!}(C_I)^{n_I}$$

$$\propto \prod_I \sum_{n_I=0}^{\infty} \frac{1}{n_I!} (C_I)^{n_I} \propto \prod_I \exp(C_I) \propto \exp\left(\sum_I C_I\right)$$

$\Rightarrow Z_1[J]$ is exponential of sum of connected diagrams

To impose $Z_1[0]=1 \Rightarrow$ omit vacuum diagrams (those w/o sources)

$$\Rightarrow Z_1[J] = \exp\left[i W_1[J]\right]$$

w/ $iW_1[J] \equiv \sum_{I \neq \{0\}} C_I$ (notation to omit vacuum diagrams)

$\Rightarrow W_1(0) = 0$

Tadpole counter term:

Compute (to $\mathcal{O}(g)$) all diagrams w/ single source and source removed

$$\langle \Omega | \phi(x) | \Omega \rangle = \frac{1}{i} \frac{\delta}{\delta J} Z_1[J]\Big|_{J=0} = \frac{\delta}{\delta J(x)} W_1[J]\Big|_{J=0}$$

$$= \frac{1}{2} i g \int d^4y \, D_F(x-y) D_F(y-y) + \mathcal{O}(g^3)$$

$$\neq 0$$

To fix this, use counterterm vertex $iY \int d^4x$ ("x" in diag)

2.21

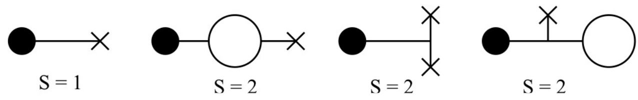


Figure 9.12: All connected diagrams with $E = 1$, $X \geq 1$ (where X is the number of one-point vertices from the linear counterterm), and $V + X \leq 3$.

Assume $Y \sim \mathcal{O}(g) \Rightarrow$ only one insertion of Y contributes

$$\langle \Omega | \varphi(x) | \Omega \rangle = \left(iY + \tfrac{1}{2} i g D_F(0) \right) \int d^4y \, D_F(x-y) + \mathcal{O}(g^3) = 0$$

$$\Rightarrow Y = \tfrac{1}{2} g D_F(0) = \tfrac{1}{2} g \underbrace{\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon}}_{I}$$

Note: this better be real since $\mathcal{L} \ni Y\varphi$ must be Hermitian

Can evaluate integral by analytic continuation to Euclidean momenta: (Wick rotation)

$$p^0 \to i p_E^0 \Rightarrow d^4p \to i\, d^4p_E \quad \text{and} \quad p^2 - m^2 + i\varepsilon \to -p_E^2 - m^2$$

$$\Rightarrow I = \int \frac{d^4p_E}{(2\pi)^4} \frac{1}{p_E^2 + m^2} = \int \frac{d\Omega}{(2\pi)^4} \int_0^\infty \frac{p_E^3 \, dp_E}{p_E^2 + m^2} \longrightarrow \infty$$

Regulate by cutting of integral at $p_E = \Lambda$

Note $\int d\Omega_4 = 2\pi^2$ and $\displaystyle\int_0^\Lambda \frac{p^3 dp}{p^2 + m^2} = \frac{1}{2}\Lambda^2 - \frac{m^2}{2} \log\left(\frac{m^2 + \Lambda^2}{m^2} \right)$

$$\Rightarrow Y = \tfrac{1}{2} g \left(\frac{1}{16\pi^2} \Lambda^2 \right)$$

dimensions $[Y] = [g] + 2[\Lambda] = 3$ ✓

"loop factor"

$\Rightarrow Y = \infty$ in just the right way to cancel ∞ generated by diag.

Can extend to higher orders $\Rightarrow$ adjusting Y like this
$\Rightarrow$ Sum over all diagrams causes cancelation between diagrams w/ Y's and those where cutting single line yields sub diagram with <u>no</u> sources (a tadpole)

$\Rightarrow$ Don't compute diagrams with tadpoles or Y's.

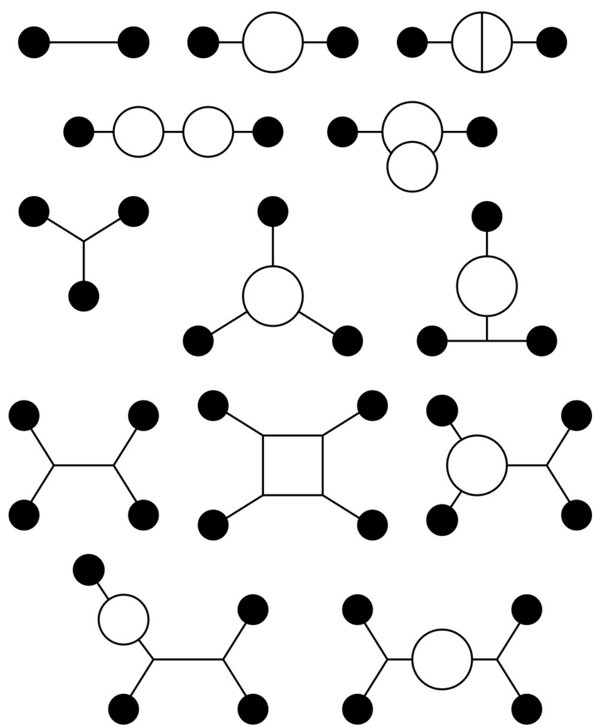


Figure 9.13: All connected diagrams without tadpoles with $E \leq 4$ and $V \leq 4$.

$\Rightarrow$ $Z[J] = \exp\left[i W[J]\right]$ w/ $W[J]$ is sum of connected diagrams with at least two sources

Now we have prescription for $\langle\Omega| T\,\varphi(x)\,|\Omega\rangle$

Define the exact Feynman propagator as

$$D_F(x_1-x_2) \equiv \langle\Omega|T\{\varphi(x_1)\,\varphi(x_2)\}|\Omega\rangle$$

Define notation $\delta_j = \frac{1}{i}\frac{\delta}{\delta J(x_j)}$ and $\varphi(x_j)=\varphi_j$

$$\Rightarrow \langle\Omega|T\varphi_1\varphi_2|\Omega\rangle = \delta_1\delta_2\, Z[J]\Big|_{J=0}$$

$$= \delta_1\delta_2\, iW[J]\Big|_{J=0} + \delta_1 iW[J]\Big|_{J=0}\delta_2 iW[J]\Big|_{J=0}$$

$$= \delta_1\delta_2\, iW[J]\Big|_{J=0}$$

where we used $\delta_j\, iW[J]\Big|_{J=0} = \langle\Omega|\varphi_j|\Omega\rangle = 0$

Think of δ_j as removing source + labeling propagator w/ endpoint x_j

Next, compute 4-pt fn

$$\langle\Omega|T\varphi_1\varphi_2\varphi_3\varphi_4|\Omega\rangle = \delta_1\delta_2\delta_3\delta_4\, Z[J]\Big|_{J=0}$$

$$= \Big[\delta_1\delta_2\delta_3\delta_4\, iW + (\delta_1\delta_2\, iW)(\delta_3\delta_4\, iW) + (\delta_1\delta_3\, iW)(\delta_2\delta_4\, iW) + (\delta_1\delta_4\, iW)(\delta_2\delta_3\, iW)\Big]\Big|_{J=0}$$

This should contribute to S-matrix via LSZ reduction $\varphi\varphi\to\varphi\varphi$

$$\langle f|S|i\rangle = i^4\int d^4x_1\, d^4x_2\, d^4x_1'\, d^4x_2'\, \exp\big(-i(k_1x_1 + k_2x_2 - k_1'x_1' - k_2'x_2')\big)$$

$$\times(\Box_1+m^2)(\Box_2+m^2)(\Box_{1'}+m^2)(\Box_{2'}+m^2)\langle\Omega|T\varphi_1\varphi_2\varphi_{1'}\varphi_{2'}|\Omega\rangle$$

Focus on a term like

$$(\delta_1 \delta_{1'} iW)(\delta_2 \delta_{2'} iW) \Rightarrow D_F(x_{11'})\, D_F(x_{22'}) \quad \text{w/ } x_{ij} = x_i - x_j$$

Let $F_{ij} = (\Box_i + m^2)(\Box_j + m^2)\, D_F(x_{ij})$

$$\Rightarrow \int d^4x_1\, d^4x_2\, d^4x_{1'}\, d^4x_{2'} \exp\left(-i\left(k_1 x_1 + k_2 x_2 - k_1' x_1' - k_2' x_2'\right)\right)$$

$$F_{11'}\, F_{22'}$$

$$= (2\pi)^4 \delta^4(k_1 - k_1')\,(2\pi)^4 \delta^4(k_2 - k_2')\, \tilde{F}(\bar{k}_{11'})\, \tilde{F}(\bar{k}_{22'})$$

w/ $\tilde{F}(k)$ FT of $F(x)$ and $\bar{k}_{ij} = (k_i + k_j)/2$

This gives no scattering: disconnected diagram

We decided to not include these $\Rightarrow$ Define connected correlation functions

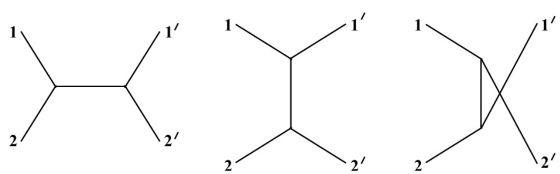
$$\langle \Omega | T \varphi_1 \cdots \varphi_N | \Omega \rangle_c = \delta_1 \cdots \delta_N\, iW[J] \Big|_{J=0}$$

$\Rightarrow$ Our $\varphi\varphi \to \varphi\varphi$ example:

$$\langle \Omega | T \varphi_1 \varphi_2 \varphi_{1'} \varphi_{2'} | \Omega \rangle_c = \delta_1 \delta_2 \delta_{1'} \delta_{2'}\, iW \Big|_{J=0}$$

4-sources + two vertices $\mathcal{O}(g^2)$

Position space Feynman diagrams "Tree-level" (no loops)



2.25

$$\Rightarrow \langle \Omega | T \varphi_1 \varphi_2 \varphi_{1'} \varphi_{2'} | \Omega \rangle_c = (ig)^2 \int d^4y \, d^4z \, D(y-z)$$

$$\times \Big[D(1-y) D(2-y) D(1'-z) D(2'-z)$$

$$+ D(1-y) D(1'-y) D(2-z) D(2'-z)$$

$$+ D(1-y) D(2'-y) D(2-z) D(1'-z) \Big] + \mathcal{O}(g^4)$$

Use fact that D_F is Green's fn

$$(\Box + m^2) D_F(x-y) = i\delta^4(x-y)$$

$\Rightarrow$ Trivial space time integrals

$$\Rightarrow \langle f | S | i \rangle = (ig)^2 \int d^4y \, d^4z \, D(y-z) \Big[\exp(-i(k_1 y + k_2 y - k_1' z - k_2' z))$$

$$+ \exp(-i(k_1 y + k_2 z - k_1' y - k_2' z)) + \exp(-i(k_1 y + k_2 z - k_1' z - k_2' y)) \Big] + \mathcal{O}(g^4)$$

Then use $D(y-z) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} e^{ik(y-z)}$

And integrate y & z to make δ-fns

$$\langle f | S | i \rangle = -ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\varepsilon} \times \Big[(2\pi)^4 \delta^4(k_1 + k_2 + k)(2\pi)^4 \delta^4(k_1' + k_2' + k)$$

$$+ (2\pi)^4 \delta^4(k_1 + k_2 + k)(2\pi)^4 \delta^4(k_2' - k_2 + k)$$

$$+ (2\pi)^4 \delta^4(k_1 - k_2' + k)(2\pi)^4 \delta^4(k_1' - k_2 + k) \Big] + \mathcal{O}(g^4)$$

$$= -ig^2 (2\pi)^4 \delta^4(k_1 + k_2 - k_1' - k_2') \times \Big[\frac{1}{(k_1 + k_2)^2 - m^2} + \frac{1}{(k_1 - k_1')^2 - m^2}$$

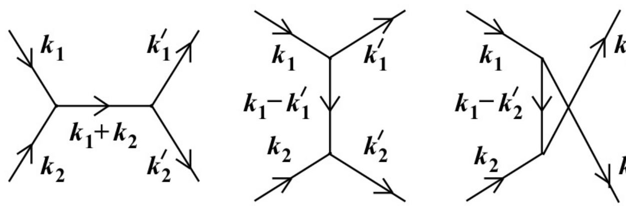
$$+ \frac{1}{(k_1 - k_2')^2 - m^2} \Big] + \mathcal{O}(g^4)$$

(Dropped $i\varepsilon$ for convinience)

Recall $\langle f | S - \mathbb{1} | i \rangle = (2\pi)^4 \delta^4(\Sigma p) \underbrace{\langle f | i\mathcal{M} | i \rangle}_{i\mathcal{M}}$

$$\Rightarrow i\mathcal{M} = -ig^2 \left[\frac{1}{(k_1+k_2)^2 - m^2} + \frac{1}{(k_1-k_1')^2 - m^2} + \frac{1}{(k_1-k_2')^2 - m^2} \right] + \mathcal{O}(g^4)$$

Momentum space Feynman diagrams



Generalizes to Feynman rules for computing $i\mathcal{M}$

1. Draw lines (called *external lines*) for each incoming and each outgoing particle.

2. Leave one end of each external line free, and attach the other to a vertex at which exactly three lines meet. Include extra *internal lines* in order to do this. In this way, draw all possible diagrams that are *topologically inequivalent.*

3. On each incoming line, draw an arrow pointing towards the vertex. On each outgoing line, draw an arrow pointing away from the vertex. On each internal line, draw an arrow with an arbitrary direction.

4. Assign each line its own four-momentum. The four-momentum of an external line should be the four-momentum of the corresponding particle.

2.27

5. Think of the four-momenta as flowing along the arrows, and conserve four-momentum at each vertex. For a tree diagram, this fixes the momenta on all the internal lines.

6. The value of a diagram consists of the following factors:

   for each external line, 1;

   for each internal line with momentum k, $+i/(k^2 + m^2 - i\epsilon)$;

   for each vertex, $iZ_g g$.

7. A diagram with L closed loops will have L internal momenta that are not fixed by rule #5. Integrate over each of these momenta ℓ_i with measure $d^4\ell_i/(2\pi)^4$.

8. A loop diagram may have some leftover symmetry factors if there are exchanges of *internal* propagators and vertices that leave the diagram unchanged; in this case, divide the value of the diagram by the symmetry factor associated with exchanges of internal propagators and vertices.

9. Include diagrams with the *counterterm vertex* that connects two propagators, each with the same four-momentum k. The value of this vertex is $-i(Ak^2 + Bm^2)$, where $A = Z_\varphi - 1$ and $B = Z_m - 1$, and each is $O(g^2)$.

10. The value of $i\mathcal{T}$ is given by a sum over the values of all these diagrams.

6. Consider a theory of three real scalar fields, A, B, and C, with the Lagrangian

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}m_A^2 A^2\\ &\frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}m_B^2 B^2\\ &\frac{1}{2}(\partial_\mu C)^2 - \frac{1}{2}m_C^2 C^2\\ &+gABC\end{aligned}$$

Write down the tree-level Feynman diagrams and corresponding scattering amplitudes for the following processes

$$AA \to AA$$
$$AA \to AB$$
$$AA \to BB$$
$$AA \to BC$$
$$AB \to AB$$
$$AB \to AC$$

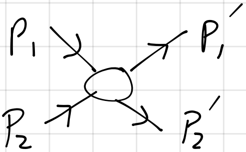
Note that your answers should be written in a form

$$\mathcal{M} = g^2\left(\frac{c_s}{m_s^2 - s} + \frac{c_t}{m_t^2 - t} + \frac{c_u}{m_u^2 - u}\right)$$

where s, t, and u are the Mandelstam variables, c_i are positive integers, and m_i^2 corresponds to m_A^2, m_B^2, or m_C^2. Note that some amplitudes could be zero.

Compute cross section for $\varphi\varphi \to \varphi\varphi$ at tree-level 12.28

First, discuss kinematics for $2 \to 2$ scattering generally



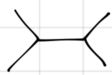
w/ all external states on-shell: $p_i^2 = m_i^2$

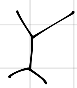
Work in CM frame: $\vec{p}_1 = \vec{p}_2 \iff \vec{p}_1'^2 - \vec{p}_2'^2 = 0$

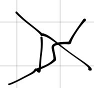
Typically choose coords w/ $\vec{p}_1 = |\vec{p}_1|\hat{z} \Rightarrow \vec{p}_2 = -|\vec{p}_1|\hat{z}$

$\Rightarrow E_1 = \sqrt{m_1^2 + |\vec{p}_1|^2}, \quad E_2 = \sqrt{m_2^2 + |\vec{p}_1|^2}$

"Mandelstam variables"

$s = (p_1 + p_2)^2 = (p_1' + p_2')^2$ 

$t = (p_1 - p_1')^2 = (p_2 - p_2')^2 = t$ 

$u = (p_1 - p_2')^2 = (p_2 - p_1')^2 = u$ 

Note $[s] = [t] = [u] = 2$

Note $s + t + u = p_1^2 + p_2^2 + 2p_1 \cdot p_2 + p_1^2 + p_1'^2 - 2p_1 \cdot p_1' + p_1^2 + p_2'^2 - 2p_1 \cdot p_2'$

$$= 3m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2\left(p_1 \cdot (-p_1 + p_1' + p_2') - p_1 \cdot p_1' - p_1 \cdot p_2'\right)$$

$$= \sum_i m_i^2$$

$\Rightarrow s, t, u$ not independent due to mom consv

Note $s = (E_1 + E_2)^2 \equiv E_{CM}^2$ the center of mass energy (squared)

Can write

2.29

$$|\vec{p}_1| = \frac{1}{2\sqrt{s}}\sqrt{s^2 - 2(m_1^2 + m_2^2)s + (m_1^2 - m_2^2)^2} \quad \text{(CM frame)}$$

$$|\vec{p}_{1'}| = \frac{1}{2\sqrt{s}}\sqrt{s^2 - 2(m_{1'}^2 + m_{2'}^2)s + (m_{1'}^2 - m_{2'}^2)^2} \quad \text{(CM Frame)}$$

Define $\cos\theta = \hat{p}_1 \cdot \hat{p}_{1'}$

$$\Rightarrow t = m_1^2 + m_{1'}^2 - 2E_1 E_{1'} + 2|\vec{p}_1||\vec{p}_{1'}|\cos\theta$$

Amplitude for $\varphi\varphi \to \varphi\varphi$ (in real scalar thy w/ $\mathcal{L}_{int} = \frac{1}{3!}g\varphi^3$)

$$\mathcal{M} = -g^2\left[\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right]$$

Recall differential cross section formula in CM frame:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{64\pi^2 E_{cm}^2}\frac{|\vec{p}_{1'}|}{|\vec{p}_1|}|\mathcal{M}|^2\,\theta(E_{cm} - m_{1'} - m_{2'})$$

Convert to form depending on s, t, u:

Take dt w/ fixed s: $dt = 2|\vec{p}_1||\vec{p}_{1'}|\, d\cos\theta$

$$= 2|\vec{p}_1||\vec{p}_{1'}|\frac{d\Omega_{CM}}{2\pi}$$

$$\Rightarrow \frac{d\sigma}{dt} = \frac{1}{64\pi s|\vec{p}_1|^2}|\mathcal{M}|^2$$

This form can be evaluated in any frame.

Note that phase space integral overcounts for identical particles by factor $n!$ for each set of identical particles

$$\Rightarrow \sigma = \frac{1}{\prod_i n_i!}\int \frac{d\sigma}{dt}\,dt \quad \text{w/ } t_{min} \text{ and } t_{max} \text{ set by } \cos\theta = \begin{cases} -1 \\ +1 \end{cases}$$

In our case, $m_c^2 = m^2 \Rightarrow |k_1'| = |k_1| = \frac{1}{2}(s-4m^2)^{1/2}$ 2.30

$$\Rightarrow t = -\frac{1}{2}(s-4m^2)(1-\cos\theta)$$

$$u = -\frac{1}{2}(s-4m^2)(1+\cos\theta)$$

Take some limits:

non rel: $s - 4m^2 \ll m^2$

$$\mathcal{M} = \frac{5g^2}{3m^2}\left[1 - \frac{8}{15}\left(\frac{s-4m^2}{m^2}\right) + \frac{5}{18}\left(1+\frac{27}{25}\cos^2\theta\right)\left(\frac{s-4m^2}{m^2}\right)^2 + \quad\right] + \mathcal{O}(g^4)$$

$\Rightarrow \frac{d\sigma}{dt}$ is nearly isotropic

Very rel: $s \gg m^2$

$$\mathcal{M} = \frac{g^2}{2\sin^2\theta}\left[3+\cos^2\theta - \left(\frac{(3+\cos^2\theta)^2}{\sin^2\theta} - 16\right)\frac{m^2}{s} + \quad\right) + \mathcal{O}(g^4)$$

$\Rightarrow$ Sharply peaked at $\theta = 0, \pi$ (forward)

To compute total cross section

$t_{min} = -(s-4m^2)$ & $t_{max} = 0$

$$\Rightarrow \sigma = \frac{g^4}{32\pi s(s-4m^2)}\left[\frac{2}{m^2} + \frac{s-4m^2}{(s-m^2)^2} - \frac{2}{s-3m^2} + \frac{4m^2}{(s-m^2)(s-2m^2)}\ln\left(\frac{s-3m^2}{m^2}\right)\right]$$
$$+ \mathcal{O}(g^6)$$

Non-rel: $\sigma = \frac{25g^4}{1152\pi m^2}\left[1 - \frac{79}{60}\left(\frac{s-4m^2}{m^2}\right) + \quad\right] + \mathcal{O}(g^6)$

rel: $\sigma = \frac{g^4}{16\pi m^2 s^2}\left[1 + \frac{7}{2}\frac{m^2}{s} + \quad\right] + \mathcal{O}(g^6)$